Descriptive Set Theory
Lecture 13

Kuratowski-Ulam theorem. This is Fubini for Baire category.

We introduce convenient notation: for a property P of points in a top space X, we write:

$\forall^* x \in X \ P(x)$ $:\Leftrightarrow$ for comeagerly many $x \in X$, $P(x)$ holds

$\exists^* x \in X \ P(x)$ $:\Leftrightarrow$ for nonmeagerly many $x \in X$, $P(x)$ holds.

Note: $\forall n \in \mathbb{N} \ \forall^* x \in X \ P_n(x) \Leftrightarrow \forall^* x \in X \ \forall n \in \mathbb{N} \ P_n(x)$.

This is a baby instance of Kuratowski-Ulam.

Theorem (Kuratowski-Ulam). Let X, Y be 2^{nd} ctbl top space (analogous to the σ-finiteness assumption in Fubini's theorem). Let $A \subseteq X \times Y$ be Baire measurable. Then:

(i) $\forall^* x$ (A_x is Baire measurable) and $\forall^* y \in Y$ (A^y is Baire meas.), where $A_x := \{y \in Y : (x,y) \in A\}$, $A^y := \{x \in X : (x,y) \in A\}$.

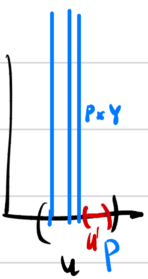
(ii) $\forall^* (x,y) \in X \times Y \ (x,y) \in A \Leftrightarrow \forall^* x \in X \ \forall^* y \in Y \ y \in A_x$
$\Leftrightarrow \forall^* x \in X$ (A_x is comeager in Y).

Similarly, with Y instead of X.

(iii) A is meager in $X \times Y \Leftrightarrow \forall^* x \in X$ (A_x is meager in Y)
$\Leftrightarrow \forall^* y \in Y$ (A^y is meager in X).

Proof. (ii) $\Leftrightarrow$ (iii). Trivial, because taking fibers commutes with complements (also unions and hence all $\Delta, \cap, \ldots$), i.e. $(A^c)_x = (A_x)^c$.

<u>**Warm up Claim 1.**</u> Let $P \subseteq X$ and $Q \subseteq Y$ be sets such that $P \times Q$ is nonmeager (resp. not nowhere dense) then P and Q are nonmeager (resp. not nowhere dense).



Proof. We prove the contrapositive, and it's enough to prove that if P is nowhere dense then $P \times Y$ is nowhere dense. Let W be a nonempty open set in $X \times Y$ and may assume it's $= U \times V$ where $U \subseteq X$ open and $V \subseteq Y$ open. $\exists\, U' \subseteq U$ nonempty open s.t. $P \cap U' = \emptyset$ hence $(P \times Y) \cap (U' \times V) = \emptyset$. $\square$

<u>**Claim 2.**</u> If $F \subseteq X \times Y$ is meager (resp. nowhere dense) then $\forall^* x \in X$ (F_x is meager (resp. nowhere dense)).

Proof. It's enough to show the nowhere dense statement. Making F bigger if necessary, we assume that F is closed. Note that $\mathrm{proj}_X : X \times Y$ is an open map by the def. of product top. By taking complements, it is enough to prove that if $W \subseteq X \times Y$ is a dense open set, then $\forall^*_x$ (W_x is dense open). The map $\begin{array}{l} Y \to X \times Y \\ y \mapsto (x,y) \end{array}$ is an embedding and W_x is the preimage of W under this map and is hence open. Fix a ctbl basis (V_n) for Y, we need to show that $\forall^*_x \forall n (W_x \cap V_n \neq \emptyset) \Leftrightarrow \forall n \forall^*_x (W_x \cap V_n \neq \emptyset)$. Fix this n, and let $U_n := \{x \in X : W_x \cap V_n \neq \emptyset\}$. We want to show this is comeager, so we'll show that U_n is open dense.

For open, note that $U_n = \mathrm{proj}_X (W \cap (X \times V_n))$, hence is open.

For density, let $U \in X$ be nonempty open. By the density of W, $W \cap (U \times V_n) \neq \emptyset$, so $\mathrm{proj}_X (W \cap (U \times V_n)) \neq \emptyset$. Hence $U_n \cap U \neq \emptyset$. □

(i) Since A is BM (Baire meas.), $A = W \Delta M$, where $W \subseteq X \times Y$ is open and $M \subseteq X \times Y$ is meager. But then $\forall x \in X$, $A_x = W_x \Delta M_x$ and W_x is open. Thus by Claim 2, $\forall^*_x \in X (A_x = W_x \Delta M_x$ and M_x is meager$)$, hence $\forall^*_x \in X (A_x$ is BM$)$. □

(iii) $\Rightarrow$. Also by Claim 2.

(iii) $\Leftarrow$. We prove the contrapositive:

$$A \text{ is nonmeager} \Rightarrow \exists^* x \, (A_x \text{ is nonmeager}).$$

Because A is BM, $A = W \triangle M$, where M is meager and W is open and nonmeager because A is nonmeager. By 2^{nd} ctblity of X and Y, W is a ctbl union of open rectangles, so one them, denoted $U \times V$, must be nonmeager. By Claim 1, both $U \subseteq X$ and $V \subseteq Y$ are nonmeager and we have:

$$\forall x \in U \, (W_x \supseteq V) \text{ hence } \forall x \in U \, (W_x \text{ is nonmeager}).$$

Thus, $\forall x \in U \, (A_x = W_x \triangle M_x$ and W_x is nonmeager$)$. Thus, $\forall x \in U \, (A_x$ is nonmeager$)$ hence $\exists^* x \in X$ $(A_x$ is nonmeager$)$. $\square$

Remark. The Baire meas. assumption on A is necessary for (ii). Indeed, there is a non-BM subset $A \subseteq \mathbb{R}^2$ that intersects every line in at most 2 points.

Applications of Kuratowski-Ulam.

Prop. Finite product of 2^{nd} ctbl Baire spaces is Baire.

Proof. Easy exercise. $\square$

The topological 0-1 law. In DST terminology, this just says that eventual equality relation is generically ergodic.

Let (X_n) be a seq. of 2^{nd} ctbl top spaces and let

$$X := \prod_n X_n.$$

We define the equivalence relation $\mathbb{E}_0(X)$ on X by: $x \, \mathbb{E}_0(X) \, y \, :\Leftrightarrow \forall^\infty n \; x(n) = y(n)$.

This is the relation of *eventual equality*.

Call a set $Y \subseteq X$ is *$\mathbb{E}_0$-invariant* if it is a union of $\mathbb{E}_0$-classes. Note that for such a set Y, and $y \in Y$, we have that $\forall n \, \forall x_0 \, \forall x_1 \ldots \forall x_n \; (x_0, x_1, \ldots, x_n, y_{n+1}, y_{n+2}, \ldots) \in Y$.

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Theorem. $\mathbb{E}_0$ is *generically ergodic*, i.e. every BM $\mathbb{E}_0$-invariant subset of X is meager or comeager.

Proof. This is a nice example of use of localization. Let Y be an $\mathbb{E}_0$-inv BM set and suppose Y is nonmeager. By the 100% lemma, $\exists$ basic open set $U \subseteq X$ s.t.

$\hat{U} \Vdash Y$. This $\hat{U}$ is of the form $\overbrace{U_0 \times U_1 \times \dots \times U_n}^{U} \times X_{n+1} \times X_{n+2} \times \dots$. Denote $X_{\leq n} := \prod_{i \leq n} X_i$ and $X_{>n} := \prod_{i > n} X_i$, so $\hat{U} = U \times X_{>n}$.

We have that $\forall^* (x,z) \in \overbrace{U \times X_{>n}}^{\hat{U}}$ $(x,z) \in Y$, by KU, $\forall^* z \in X_{>n}\ \forall^* x \in U\ (x,z) \in Y$. Thus $\forall^* z \in X_{>n}$ $\exists x \in U\ (x,z) \in Y$. But by E_0-invariance, $\forall x \in X_{\leq n}\ (x,z) \in Y$. Thus, $\forall^* z \in X_{>n}\ \forall x \in X_{\leq n}$ $(x,z) \in Y$. By KU again, $\forall^* (x,z) \in X\ (x,z) \in Y$, so Y is comeager. $\square$

Theorem. Let X be a nonempty perfect Polish space (e.g. $\mathbb{R}$). There is no Baire meas. well-ordering of X, i.e. $\exists$ well-ordering $<$ of X that is BM as a subset of X^2.

Proof. Suppose $<$ is such a well-ordering. An initial segment is a subset of X closed downward under $<$.

Claim. If $A \subseteq X$ is nonmeager BM initial segment, then $(< |_A) \subseteq X^2$ is nonmeager.

Proof.